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ON THE TRANSITION TO TRANSVERSE ROLLS IN AN INFINITE VERTICAL FLUID LAYER—A POWER SERIES SOLUTION

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Abstract—The criteria governing the transition from the conduction regime to the transverse roll regime in a vertical infinite fluid layer are found by means of a power series method. The calculated values are shown to be in agreement with previously published results. Findings concerning the variation of the Gr_c and α_c with Pr are presented. An anomalous behaviour was found at $Pr \simeq 1.75$.

NOMENCLATURE

A_{mn}, boundary condition matrix;
a_n, b_n, c_n, d_n, Galerkin method constants;
b_k, c_k, power series method constants;
B_{kn}, C_{kn}, particular solution vectors;
D, d/dz;
g, acceleration of gravity;
F, G, C_n, S_n, Galerkin method functions;
Gr, Grashof number, $g\beta\Delta TH^3/\nu^2$;
G_n, general solution vector;
H, spacing between plates;
N, number of terms in a series;
p, even part of θ;
Pr, Prandtl number, ν/κ;
q, odd part of θ;
x, y, z, spatial coordinates;
s, even part of W;
t, odd part of W;
T, temperature;
T_0, reference temperature;
u, v, w, velocities;
W, functional dependence of W on z.

Greek symbols

α, spatial wavenumber;
β, thermal expansion coefficient;
ΔT, temperature difference of plates;
θ, functional dependence of T on z;
κ, thermal diffusivity;
$\lambda_n, \mu_n, \psi_n, \omega_n$, Galerkin method constants;
ν, kinematic viscosity;
ψ, streamfunction.

Subscript

c, critical condition.

INTRODUCTION

THIS paper deals with the stability of the conduction regime in a vertical infinite layer of fluid. Figure 1 shows the problem configuration. The fluid is contained between two parallel, flat, isothermal, rigid plates of essentially infinite area, separated by distance H, and situated parallel to the gravity vector, g. The temperature difference of the plates is ΔT; one plate having the temperature $T_0 - \Delta T/2$ and the other plate the temperature $T_0 + \Delta T/2$. A Cartesian co-ordinate system with its origin mid-way between the plates is constructed such that x is parallel to gravity, y is across the plates and z is perpendicular to the plates. The corresponding fluid velocities are u, v and w. It is well known that, provided the Grashof number ($Gr = g\beta\Delta TH^3/\nu^2$) is larger than zero, an imbalance between pressure and gravity forces causes fluid motion to always exist. At low Gr, this motion is described as a base flow, being cubic in z and independent of x and y. The corresponding temperature is linear in z and heat transfer is by conduction only (hence the regime name). At sufficiently large Gr, the conduction regime becomes unstable and suffers a transition to a multicellular convection regime. This transition is the subject of the present paper.

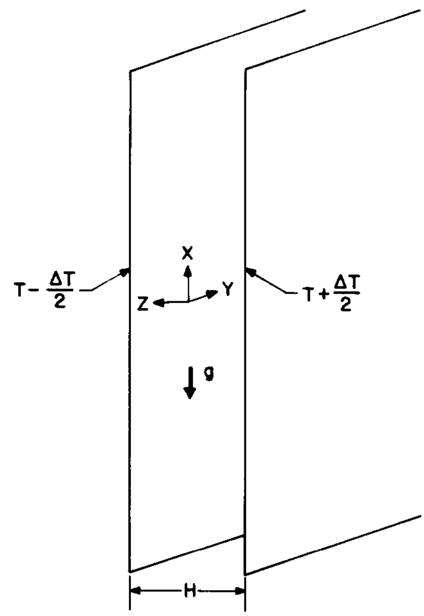


FIG. 1. Problem configuration.

It is generally accepted that the multicellular convection regime, at least near the point of transition from the conduction regime, is a combination of the base flow and rolls whose axes are in the y direction. These rolls are termed transverse rolls and are independent of the y position. This paper deals only with transitions to steady rolls. For the stability analysis, the variables w and temperature, T, are therefore assumed to have the forms:

$$w = W(z)\exp(i\alpha x), \tag{1a}$$

and

$$T = \theta(z)\exp(i\alpha x), \tag{1b}$$

where i is the imaginary constant and α is the spatial wavenumber. Following arguments such as those presented by Korpela *et al.* [1], the following equations, which govern the transition point, are obtained:

$$(\mathrm{D}^2-\alpha^2)^2 W - i\alpha Gr\,\mathrm{D}\theta + \left| i\alpha^3 \frac{Gr}{6}\left(z^3-\frac{z}{4}\right) + i\alpha Grz \right| W - i\alpha\frac{Gr}{6}\left(z^3-\frac{z}{4}\right)\mathrm{D}^2 W = 0, \tag{2a}$$

and

$$(\mathrm{D}^2-\alpha^2) + PrW - i\alpha Pr\frac{Gr}{6}\left(z^3-\frac{z}{4}\right)\theta = 0, \tag{2b}$$

with the boundary conditions

$$\theta = W = \mathrm{D}W = 0 \text{ at } z = \pm 1/2. \tag{2c}$$

In the above equations, $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}z$, z is nondimensionalized with H, W with ν/H and T with ΔT. Pr is the Prandtl number, ν/κ.

The first detailed treatment of this transition problem was by Rudakov [2]. Following the earlier work of Birikh [3], who treated the problem as an isothermal flow, Rudakov determined the stability condition using the Galerkin method. Rudakov concluded that the critical Grashof number, Gr_c, (i.e., the Gr at which the transition to the transverse roll regime occurs) was a function of Pr, ranging over $7360 \leqslant Gr_c \leqslant 8160$† for $0 \leqslant Pr \leqslant 10$. The critical wavenumber was found to be only weakly dependent on Pr, having the approximate value $\alpha_c = 2.8$. Vest and Arpaci [4] found experimentally that for air $Gr_c = 8700 \pm 10\%$ and $\alpha_c = 2.74$. Gersuni and Zhukovitskii [5] have presented tabular results of Gr_c and α_c in their text on convective stability. For $Pr = 0.2$, 1 and 5 they found $Gr_c = 7520$, 7952 and 7840, and $\alpha_c = 2.70$, 2.84 and 2.80, respectively. Korpela *et al.* [1], also using a Galerkin method, calculated the stability criterion over a range of Prs. Gr_c was found to be Pr dependent, with tabulated results given for $\alpha_c = 2.65$. For $Pr = 0$, Gr_c and α_c were found to be 7930.1 and 2.70, respectively. The α_c was shown to increase to 2.8 at $Pr = 12.7$. A point of similarity between the theoretical results of the various workers was that for $Pr > 1$ more terms were required in the Galerkin approximations in order to achieve convergence.

The present paper reports on a study initiated for two purposes:

(a) To find an alternate solution technique to the Galerkin method in order to simplify the stability analysis; and

(b) to calculate highly converged values of Gr_c and α_c and investigate their behaviour over a range of Pr.

The motivation for the first purpose may be found by inspecting equations (2). These equations are a system of ordinary differential equations with variable coefficients. As such, they are solvable by the power series method without the necessity of resorting to a Galerkin method. The second purpose is motivated on two counts, first as a means of checking the success of the first purpose, and second for its intrinsic scientific value.

THE POWER SERIES METHOD

Many thermofluid stability problems to date (a notable exception being the horizontal (Bénard) problem (Pellew and Southwell [6]) and its extension to longitudinal rolls (Birikh *et al.* [7]) for which simple exponential solutions are valid) have been solved using the Galerkin technique. This method involves choosing functional forms (which satisfy the boundary conditions) for W and θ, expressing W and θ as series of these functional forms, substituting these expressions for W and θ into equations (2), orthogonalizing the equations with respect to the functional forms, and iteratively solving the resultant set of homogeneous equations by choosing Pr and α and finding the Gr which makes the determinant of the equations zero. The usual functional forms are those due to Harris and Reid [8]:

$$W \text{ or } \phi\dagger = \sum_{n=1}^{\infty} a_n C_n(z) + ib_n S_n(z), \tag{3a}$$

and

$$\theta = \sum_{n=1}^{\infty} d_n \sin_n(\omega_n z) + ie_n \cos(\psi_n z), \tag{3b}$$

where

$$C_n(z) = \frac{\cosh(\lambda_n z)}{\cosh(\frac{1}{2}\lambda_n)} - \frac{\cos(\lambda_n z)}{\cos(\frac{1}{2}\lambda_n)}, \tag{4a}$$

and

$$S_n(z) = \frac{\sinh(\mu_n z)}{\sinh(\frac{1}{2}\mu_n)} - \frac{\sin(\mu_n z)}{\sin(\frac{1}{2}\mu_n)}, \tag{4b}$$

†All values of Gr and α have been transformed to agree with the nondimensionalizing scheme adopted in the present paper.

†Most investigators use the stream function ϕ rather than W.

and λ_n, μ_n, ω_n and ψ_n are chosen to satisfy the four boundary conditions on W and the two boundary conditions on θ respectively. The orthogonalization of equations (2), after substitution of equations (3) and (4), is not straightforward as it involves the integration of complicated functions, some involving multiples of hyperbolic, trigonometric and z^m terms. This complexity makes the computations involving the homogeneous equations algebraically cumbersome. In addition, previous investigators solved from 24th to 40th order determinants in order to determine the Gr_c to 4 or 5 figures. The complexity and computer time required by the Galerkin method provided the impetus for the search for a simpler technique.

The system of equations involved in this problem should be tractable by the power series method. This method, which has found great success in such cases as Bessel's equations, is the classical method for solving ordinary differential equations with variable coefficients. To the author's knowledge, the only other application of the power series method to a thermofluid stability problem was by Sparrow *et al.* [9], where they treated the effect of a nonlinear temperature profile on the stability criteria in a horizontal fluid layer. The present technique of applying the method is, of necessity, more sophisticated, since the present problem is considerably more complicated. The method depends upon expanding the variables in simple power series, substituting into the equations, equating the collected coefficients of like powers of the independent variable to zero, and thereby discovering a recursion formula for the coefficients of the series. A number of coefficients remain arbitrary (later chosen to satisfy boundary conditions) and the recursion formula is used to write all coefficients in terms of the arbitrary coefficients. This results in a solution involving only the arbitrary coefficients multiplied by power series of the independent variable. Classical power series solutions involved finding quite simple recursion formulae which were suitable to manual manipulation. Very complicated equations were not tractable to large numbers of terms because the associated series did not lend themselves to finding explicit recursion formulae. This requirement for explicit definitions of recursion formulae is no longer necessary, at least when the problem is computer solved. Below, a technique is described for solving the vertical layer problem by means of the power series method. Although the technique is demonstrated for this particular problem, it should be applicable to many linear stability problems.

In the power series method, variables W and θ are assumed to satisfy the series

$$W = \sum_{k=1}^{\infty} b_k z^{k-1}, \tag{5a}$$

and

$$\theta = \sum_{k=1}^{\infty} c_k z^{k-1}, \tag{5b}$$

where, in general, the coefficients b_k and c_k are complex constants. These equations are differentiated and substituted into equations (2a) and (2b). On collecting terms of equal powers of z and setting these collections to zero, the following equations result:

$$b_1 \equiv b_2 \equiv b_3 \equiv b_4 \equiv c_1 \equiv c_2 \equiv \text{arbitrary}, \tag{6a}$$

for $k > 4$,

$$b_k = \frac{1}{(k-1)(k-2)(k-3)(k-4)} \left\{ 2\alpha^2(k-3)(k-4)b_{k-2} - \alpha^4 b_{k-4} + i\alpha Gr(k-4)c_{k-3} - Gr\left[\frac{i\alpha^3}{6} b_{k-7}\Delta_{1,k-7} - \left(\frac{i\alpha^3}{24} - i\alpha + \frac{i\alpha}{6}(k-6)(k-7)\right) b_{k-5}\Delta_{1,k-5} + \frac{i\alpha}{24}(k-4)(k-5)b_{k-3}\right]\right\}, \tag{6b}$$

and for $k > 2$:

$$c_k = \frac{1}{(k-1)(k-2)}\left[\alpha^2 c_{k-2} - Pr b_{k-2} + i\alpha Pr \frac{Gr}{6} c_{k-5}\Delta_{1,k-5} - i\alpha Pr \frac{Gr}{24} c_{k-3}\Delta_{1,k-3}\right], \tag{6c}$$

where

$$\Delta_{nm} = 0, \quad m < n, \tag{6d}$$

and

$$\Delta_{nm} = 1, \quad m \geqslant n. \tag{6e}$$

It may be seen therefore, that each higher order constant is expressed in terms of lower order constants, which in turn may be expressed in terms of even lower order constants, and ultimately all constants may be expressed in terms of the arbitrary constants.

The solution of the stability problem depends on finding b_1, b_2, b_3, b_4, c_1 and c_2. These six constants may be expressed in the form of a general solution vector

$$G_n = (b_1, b_2, b_3, b_4, c_1, c_2). \tag{7}$$

All the constants may then be expressed in terms of particular solution vectors, B_{kn} for the b_k constants and C_{kn} for the c_k constants, such that

$$b_k = B_{kn}G_n, \tag{8a}$$

and

$$c_k = C_{kn}G_n, \tag{8b}$$

where the summation on repeated indices convection applies. For example,

$$b_1 = (1,0,0,0,0,0)\cdot(b_1, b_2, b_3, b_4, c_1, c_2) \tag{9a}$$

or

$$B_{1n} = (1,0,0,0,0,0), \tag{9b}$$

while

$$c_1 = (0, 0, 0, 0, 1, 0) \cdot (b_1, b_2, b_3, b_4, c_1, c_2) \quad (9c)$$

or

$$C_{1n} = (0, 0, 0, 0, 1, 0). \quad (9d)$$

It follows that the first four B_{km} and the first two C_{kn} vectors are unit vectors, for $k > 4$,

$$B_{kn} = \frac{1}{(k-1)(k-2)(k-3)(k-4)} \times \Bigg\{ 2\alpha^2(k-3)(k-4)B_{k-2,n} - \alpha^4 B_{k-4,n} + i\alpha Gr(k-4)C_{k-3,n} - Gr \Bigg| \frac{i\alpha^3}{6} B_{k-7,n}\Delta_{1,k-7} - \left(\frac{i\alpha^3}{24} - i\alpha + \frac{i\alpha}{6}(k-6)(k-7)\right) B_{k-5,n}\Delta_{1,k-5} + \frac{i\alpha}{24}(k-4)(k-5)B_{k-3,n}\Delta_{1,k-3} \Bigg| \Bigg\}, \quad (10a)$$

and for $k > 2$,

$$C_{kn} = \frac{1}{(k-1)(k-2)} \Bigg| \alpha^2 C_{k-2,n} - PrB_{k-2,n} + i\alpha Pr \frac{Gr}{6} C_{k-5,n}\Delta_{1,k-5} - i\alpha Pr \frac{Gr}{24} C_{k-3,n}\Delta_{1,k-3} \Bigg|. \quad (10b)$$

The B_{kn} and C_{kn} may therefore be found independently of knowing G_n.

By means of equations (8), equations (5) may be rewritten as

$$W = \sum_{k=1}^{\infty} B_{kn} G_n z^{k-1}, \quad (11a)$$

and

$$\theta = \sum_{k=1}^{\infty} C_{kn} G_n z^{k-1}. \quad (11b)$$

The six homogeneous boundary conditions (equations (2c)) may now be used to generate a set of six homogeneous equations for the six elements of the vector G_n. A solution exists if and only if the determinant of this set is zero. The elements of the determinant are given by

$$A_{1n} = \sum_{k=1}^{\infty} B_{kn}(1/2)^{k-1}, \quad (12a)$$

$$A_{2n} = \sum_{k=1}^{\infty} B_{kn}(-1/2)^{k-1}, \quad (12b)$$

$$A_{3n} = \sum_{k=1}^{\infty} B_{kn}(k-1)(1/2)^{k-1}, \quad (12c)$$

$$A_{4n} = \sum_{k=1}^{\infty} B_{kn}(k-1)(-1/2)^{k-1}, \quad (12d)$$

$$A_{5n} = \sum_{k=1}^{\infty} C_{kn}(1/2)^{k-1}, \quad (12e)$$

$$A_{6n} = \sum_{k=1}^{\infty} C_{kn}(-1/2)^{k-1}. \quad (12f)$$

The solution of the stability problem is now straightforward. For particular α and Pr, a Gr is assumed, B_{kn} and C_{kn} are found, the A_{mn} are found, $|A_{mn}|$ is determined, a correction on Gr is calculated and the procedure is repeated until the determinant is zero to within some prescribed limit. This procedure may be used to search for the α that results in the lowest value of Gr. The corresponding α and Gr are then α_c and Gr_c.

COMPARISON OF THE POWER SERIES METHOD WITH THE GALERKIN METHOD OF PREVIOUS WORKERS

The power series method was used to obtain solutions which could be compared directly with results of Korpela *et al.* [2]. The data for $\alpha = 2.65$ and various Pr's are presented in Table 1 for a range of number of terms, N, used in the power series. In most cases Gr was iterated, using a Newton–Raphson technique, until the change in Gr, due to the error in $|A_{mn}| = 0$, was less than the fraction 10^{-8}. Generally, 4 figure exactness in Gr was attained with $N = 50$, for $Pr \leqslant 1.0$. Convergence in Gr to 7 figures required a maximum of 70 terms in these cases. For $Pr = 10.0$, the required N was significantly increased, a difficulty noted by previous workers. As well, convergence to 7 figures was difficult to attain, the sixth figure often varying. In this case, the Gr was iterated only until the fractional change was 10^{-6}.

Table 1. Convergence of Gr for $\alpha = 2.65$

Pr	$N =$	10	20	30	40	50	60	70	80
0.001		5255.860	6332.856	8633.156	7913.691	7919.772	7919.743	7919.743	7919.743
0.01		5214.586	6296.344	8385.500	7805.845	7811.077	7811.053	7811.053	7811.053
0.1		4920.174	6020.062	7705.900	7351.725	7355.020	7355.006	7355.006	7355.006
1.0		—	—	—	7955.360	7989.764	7989.525	7989.525	7989.525
	$N =$	120	130	140	150	160			
10.0		—	7873.02	7873.42	7898.23	7898.23			

Table 2 shows a comparison between the present results and those of Korpela *et al.* [1]. Except in the

Table 2. Comparison with Korpela *et al.* [1]

Pr	α	Present	Korpela *et al.*
0.001	2.65	7919.743	7920
0.01	2.65	7811.053	7811
0.1	2.65	7355.006	7355
1.0	2.65	7989.525	7989
10.0	2.65	7898.23	7898

case of $Pr = 1.0$, the present results agree to the exactness given by those workers. The disagreement for $Pr = 1.0$ is in the last place only (difference of $< 10^{-2}\%$). This excellent agreement was interpreted as validation of the power series method.

The application of the Galerkin method to the problem at hand results in the evaluation of 56 (see Vest [10]) integrals of the form

$$\int_{-1/2}^{1/2} z^m FG\,dz,$$

where F and G are either trigonometric or hyperbolic functions. These integrals may be evaluated exactly, however, the procedure is tedious with the results being, in some cases, complicated formulae. An alternative is numerical integration, however, since the functions F and G contain the constants λ_n, μ_n, ω_n and ψ_n (as can be seen from equations (4)), the numerical integrations must be performed for all products $z^m FG$ which are not known to be orthogonal. If N terms are used in the approximating functions, at the very least 56 N integrals must be evaluated. These integrals may then be stored in the computer for recall when needed. However, a sophisticated bookkeeping system is needed to ensure that the proper integral is inserted at the proper place in the orthogonalized equations. Contrast the above procedure with the programming steps in the power series method. After defining the six unit vectors (B_{1n}, B_{2n}, B_{3n}, B_{4n}, C_{1n}, C_{2n}) equations (10) may be solved sequentially for the remaining B_{kn} and C_{kn}. The matrix of the equations, A_{mn}, may then be found from equations (13). Both steps may be accomplished by using a single pair of nested do-loops. The simplicity of the programming required is manifest in the fact that the actual solution part of the program (not considering $I/0$, determinant solvers, etc.) required only 40 assignment statements of which only 5 required more than 5 mathematical operations.

The computer time required to solve the set of homogeneous equations is essentially controlled, in the Galerkin method, by the time required to solve the determinant. In the power series method, the determinant is always 6th order, a determinant which requires little time to solve, and the speed of solution is governed by the time required to find the B_{kn}, C_{kn} and A_{mn}. Generally, 50 terms were required in the power series method to obtain the same exactness as Korpela *et al.* achieved with 7 terms (hence a 28th order determinant). Computer time to solve a determinant increases as the determinant order raised to a power between 2 and 3. Increasing the exactness of the solution requires increasing the order of the determinant; therefore, in a Galerkin solution, computer time would increase rapidly if more exactness was required. The calculation of 50 terms in the power series method required generation of 606 algebraically simple terms (100 specific solution vectors of six terms each plus the six terms of A_{mn}). Computer time to incorporate more terms in the series increases linearly with N (an indication of how exactness increases with N is gained by considering that, for $Pr < 1.0$, convergence in Gr_c to 12 or 13 figures was obtained with $N < 90$). The set up and solution of one determinant by the power series method required approximately 0.0064 s/term (using a CDC 6400, Cyber 172 computer). The total time required to solve fcr Gr_c and α_c at 60 values of Pr ($0.00001 \leqslant Pr \leqslant 10$) was 781 s, just over 13 min. This time included the search time to find the α which produced a minimum in Gr for a particular Pr.

THE STABILITY RESULTS

The power series method was used to determine essentially exact values of the stability condition implied by equations (2). Values of Gr_c and α_c were calculated for 60 values of Pr in the range $0.00001 \leqslant Pr \leqslant 10.0$ and separately for the limiting case $Pr = 0$. It was found that in general, Gr_c had to be calculated to 7 figures in order to find α_c to 4 figures. Sample values are given in the Appendix, with the results being summarized in Fig. 2. Also plotted in Fig. 2 are selected results from Rudakov [2], and the tabulated results of Gershuni and Zhukovitskii [5] and Korpela *et al.* [1]. Generally, the agreement is good. The disagreement for $Pr > 1$ is probably attributable to the use of too few terms in the Galerkin approximation by previous workers. The experimental result of Vest and Arpaci [4], $Gr_c = 8700 \pm 10\%$, agrees with the present value (for air) of approximately 8038. The agreement between the present α_c values and those of previous workers is fair. Rudakov states that $\alpha_c \simeq 2.8$ and a constant, while Korpela *et al.* show α_c to vary from 2.7 to 2.8. In actuality, α_c varies in a fairly complicated manner, from a low of 2.688 to a high of 2.812. The α_c values of Gershuni and Zhukovitskii agree with the present results to within $\sim 1\%$. The value of α_c found experimentally for air by Vest and Arpaci was 2.74, which differs by only 2.5% from the presently predicted value of 2.810. Since α_c and Gr_c are interdependent, the disagreement in α_c, between the present results and those of some previous works, undoubtedly accounts for some of the disagreement in Gr_c.

The limiting case of $Pr = 0$ was solved separately by taking the limiting form of equations (2) and programming a separate solution. Gr_c and α_c were found to have the values 7930.055 and 2.688 respectively. These results are in essential agreement with the results of Korpela *et al.* [1].

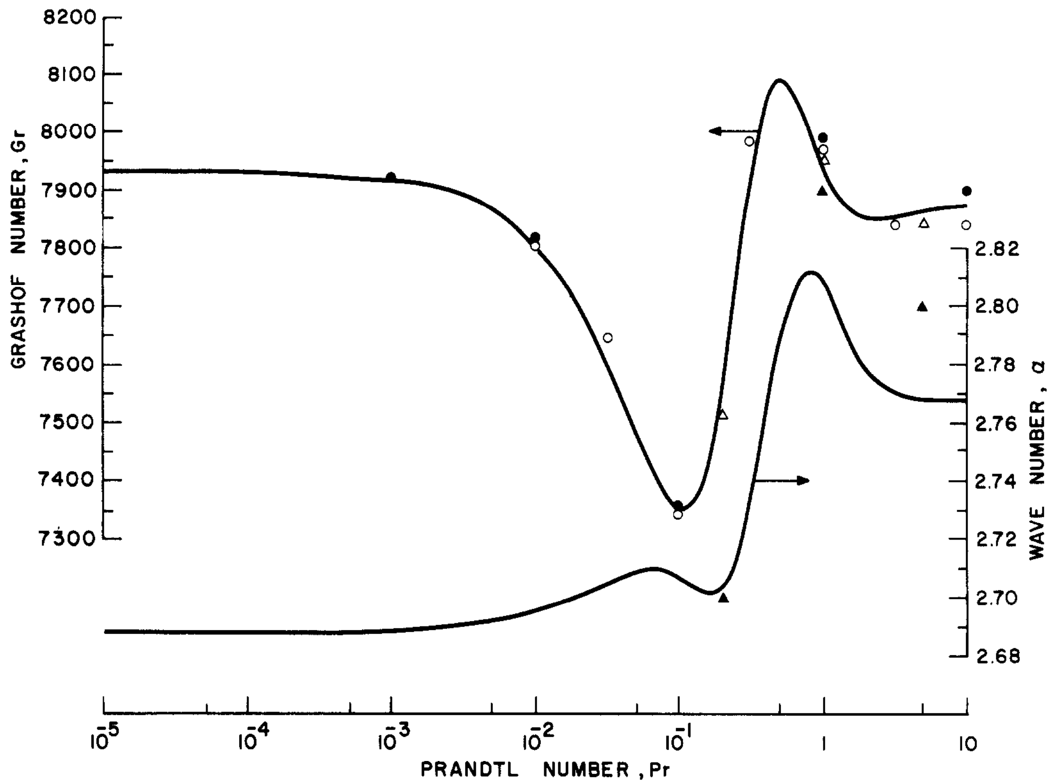


FIG. 2. The critical conditions (—— Present results; ● Gr_c [2]; △ Gr_c [5]; ▲ α_c [5]; ○ Gr_c [1]).

BEHAVIOUR OF Gr_c AND a_c WITH Pr

A great advantage of the power series method is the ease with which the resultant series may be manipulated. Using this advantage, a preliminary investigation into the nature of the Gr_c and α_c dependence on Pr was performed. The constants of the power series are in general complex numbers. For the vertical layer, the solution is a combination of even and odd terms. Since the solution is only obtainable to within an arbitrary constant (the equations are homogeneous), c_3 was given the value 1.0 to make the even part of the solution real. If the solutions for θ and W are denoted by

$$\theta = p + iq, \tag{13a}$$

and

$$W = s + it, \tag{13b}$$

then the real part of T and W are given by equations (1) as

$$Re(T) = p\cos\alpha r_1 - q\sin\alpha r_1, \tag{14a}$$

and

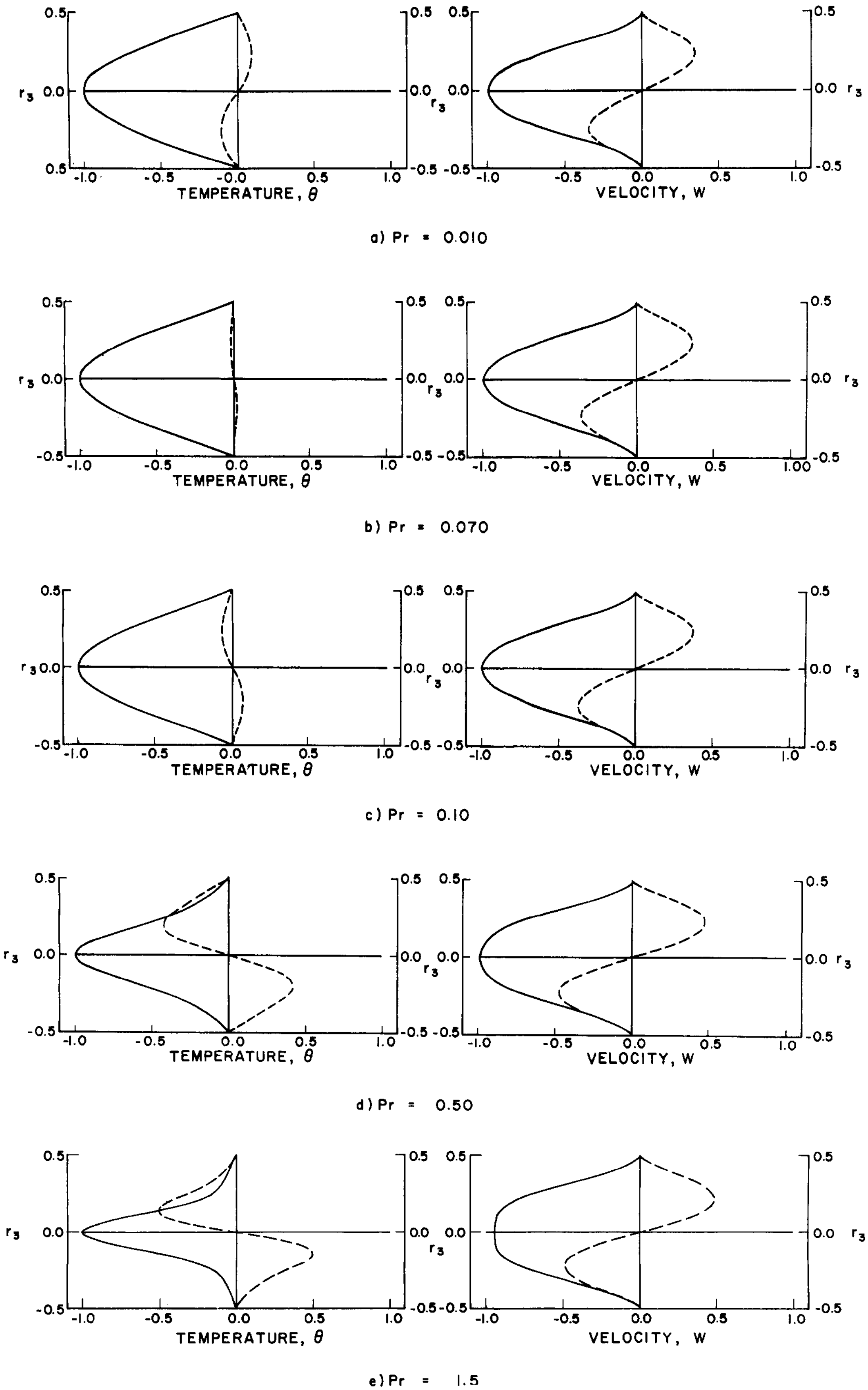
$$Re(W) = s\cos\alpha r_1 - t\sin\alpha r_1, \tag{14b}$$

where p and s are even solutions and q and t are odd.

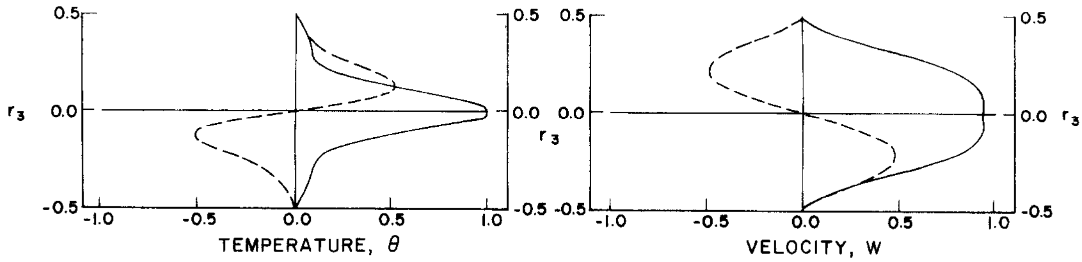
Figure 2 indicates four separate regions of Gr_c dependence on Pr. The first region extends over the range $0 < Pr \leqslant 0.1$, and is characterized by a decrease of Gr_c with Pr. The second region extends over the range $0.1 < Pr \leqslant 0.5$, and is characterized by a sharp increase of Gr_c with Pr. The third region extends over the range $0.5 < Pr \leqslant 2.3$, and is characterized by a sharp decrease in Gr_c with Pr. The final region extends from $Pr = 2.3$ to $Pr = 10.0$ (and presumably infinity) for which Gr_c increases slowly with Pr and appears to be approaching an asymptotic value.† Representative profiles of p, q, s and t were calculated for each region and are presented in Fig. 3. The profiles are normalized with the moduli of equations (13a) and (13b). The magnitudes and ratio of the normalization factors for θ and W are plotted in Fig. 4.

With reference to Fig. 3, the following points are considered. For $Pr = 0.01$, q and t are smaller than p and s respectively, and are in phase with each other. However, at $Pr = 0.07$, q has suffered a reversal now being out of phase with t. At $Pr = 0.1$, the relative magnitude of q has increased. The decrease in Gr_c with Pr is therefore associated with a decrease in the relative magnitude of q. The minima at $Pr = 0.1$ is probably associated with the reversal in the phase of q. At $Pr = 0.5$, noticeable changes in the shape of p are manifest, p being thinned and taking on a much more peaked appearance. In addition s has thickened. The increase in Gr_c with Pr is therefore associated with an increase in the magnitude of q.

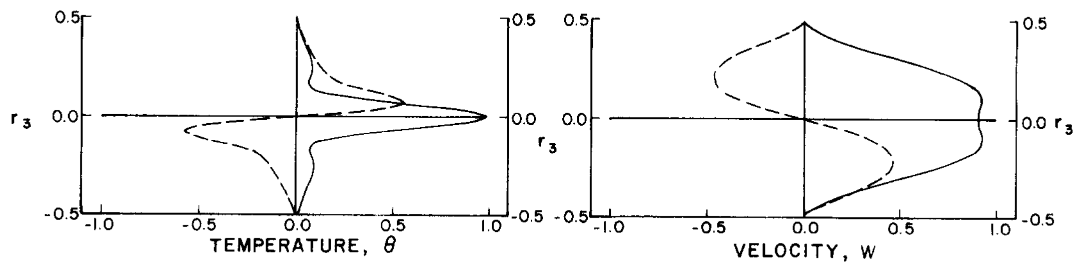
†It is worth noting here that, for $Pr > 12.7$, Korpela *et al.* have shown that the transition to convection results in an unsteady motion. Transition values for steady motions are therefore of little interest for $Pr > 12.7$.



3. Profiles for θ and W (—— p and s, ------ q and t).



f) Pr = 2.0



g) Pr = 10.0

FIG. 3 (continued).

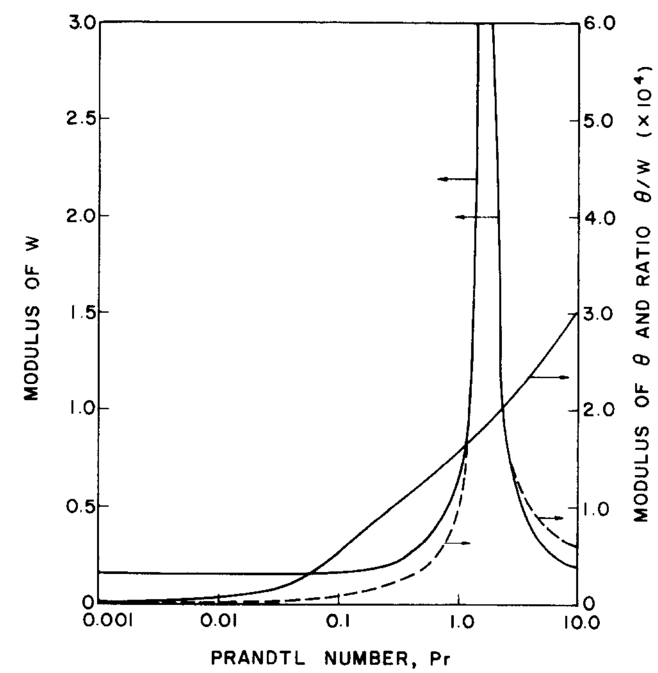


FIG. 4. Scales for θ and W (——Ratio θ/W).

The maxima at $Pr = 0.5$ is probably associated with the change in shape of the profiles of both p and s. At $Pr = 1.5$, s has become blunt nosed while p has become very pointed. A reversal in sign of all profiles occurs between $Pr = 1.5$ and $Pr = 2.0$. The decrease in Gr_c with Pr is therefore associated with a continued change in profile shape. The minima at $Pr = 2.0$ is probably associated with the sign reversal of the profiles. For $Pr = 10.0$, p has become clearly trimodal and s has become bimodal. This plot plainly shows why convergence deteriorates for $Pr > 1$; the curves become so complex as to require a large number of terms to approximate their shapes.

Figure 4 shows the behaviour of the scaling factors for θ and W and the ratio of these factors (θ/W) for a range of Pr. It must be kept in mind that the absolute magnitude of the scale factors has no direct physical significance. The magnitude is a purely mathematical result caused by arbitrarily setting c_3 to 1. However, it is probably significant that the discontinuity observed at $Pr \simeq 1.75$, in both the θ and V_3 curves, is the same point at which the profiles reverse direction. Although the individual scale factors exhibit discontinuities, the ratio θ/W is well behaved (as would be expected since here the arbitrarity of the scales would cancel out). It must be reemphasized that the scales have arbitrary magnitudes and no further conclusions as to the significance of the discontinuities should be made except to suggest that a non-linear analysis of the problem, to determine actual magnitudes, and collection of experimental data at $Pr \simeq 1.75$, may yield interesting results.

In general, it may be said that the variation of α_c with Pr parallels that of Gr_c with Pr. The observation that extrema of the two curves in Fig. 2 do not exactly coincide, and that there is some variation in behaviour, is probably due to that fact that α enters equations (2) with various powers (up to α^4), whereas Gr enters only with the power 1.

The power series solution may also be used to generate profiles of the derivatives of θ and W. Since power series are easily manipulated by digital computers, these profiles may be calculated simply and quickly.

CONCLUSIONS

The conclusions of the present study are as follows:

1. The power series method offers a simple, accurate and fast alternative to the Galerkin method for the solution of stability problems. Due to the programming ease of the power series method, it should be considered as the primary technique for solving stability problems involving equations with non-constant coefficients. More complicated techniques should be used only if this method proves inadequate.

2. By means of the power series method, essentially exact solutions of Gr_c and α_c, as functions of Pr, have been found for the vertical layer problem.

3. The variation of the Gr_c and α_c data with Pr has been associated with various changes in the relative importance and sign of the even and odd parts of T and W.

4. A very interesting discontinuity in the calculated "magnitudes" of θ and W suggests more theoretical and experimental work should be done for fluids with $Pr \simeq 1.75$.

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APPENDIX

Critical data

Pr	α_c	Gr_c
0.00001	2.688	7929.923
0.00005	2.688	7929.394
0.0001	2.688	7928.733
0.0003	2.688	7926.094
0.0005	2.689	7923.463
0.001	2.689	7916.920
0.003	2.690	7891.247
0.005	2.692	7866.357
0.01	2.695	7807.448
0.02	2.699	7703.050
0.03	2.704	7615.100
0.05	2.708	7482.533
0.07	2.709	7399.389
0.10	2.707	7349.516
0.30	2.729	7896.585
0.50	2.789	8093.723
0.70	2.810	8041.422
1.0	2.808	7940.235
1.5	2.790	7866.340
2.0	2.779	7850.043
2.3	2.775	7848.831
3.0	2.771	7852.483
5.0	2.768	7863.939
7.0	2.767	7868.426
10.0	2.767	7870.43

SUR LA TRANSITION VERS LES ROULEAUX TRANSVERSES DANS UNE COUCHE FLUIDE VERTICALE ET INFINIE—UNE SOLUTION EN SERIE DE PUISSANCE

Résumé—On trouve, à l'aide d'une méthode en série de puissance, le critère qui gouverne la transition entre le régime de conduction et le régime des rouleaux transverses dans une couche fluide infinie et verticale. Les valeurs calculées sont en bon agrément avec les résultats précédemment publiés. On présente les résultats concernant la variation de Gr_c et de α_c en fonction de Pr. On trouve un comportement anormal à $Pr = 1{,}75$.

О ПЕРЕХОДЕ К ПОПЕРЕЧНЫМ ВАЛАМ В БЕСКОНЕЧНОМ ВЕРТИКАЛЬНОМ СЛОЕ ЖИДКОСТИ. РЕШЕНИЕ МЕТОДОМ СТЕПЕННЫХ РЯДОВ

Аннотация — Методом степенных рядов найден критерий, определяющий переход от режима теплопроводности к режиму поперечных валов в вертикальном бесконечном слое жидкости. Показано, что рассчитанные значения согласуются с ранее опубликованными данными. Представлены результаты по зависимости Gr_c и α_c от Pr. Обнаружено аномальное поведение при $Pr \simeq 1{,}75$.

ÜBER DEN ÜBERGANG ZU QUERLAUFENDEN WALZEN IN EINER UNENDLICH AUSGEDEHNTEN VERTIKALEN FLUIDSCHICHT—EINE POTENZREIHENLÖSUNG

Zusammenfassung—Mittels einer Potenzreihenmethode wird das Kriterium entwicklet, das in einer unendlich ausgedhnten vertikalen Fluidschicht den Übergang von den Gesetzmäßigkeiten der Wärmeleitung zu den Gesetzmäßigkeiten der querlaufenden Walze beschreibt. Es wird gezeigt, daß die berechneten Werte mit früher veröffentlichten Ergebnissen übereinstimmen. Erkenntnisse über die Änderung von Gr_c und α_c mit Pr werden dargestellt. Bei $Pr \simeq 1{,}75$ wurde ein ungewöhnliches Verhalten festgestellt.